



# Influence of the Long-Range Dependence in Rainfall in Modelling Oueme River Basin (Benin, West Africa)

Eliezer Iboukoun Biao<sup>1, \*</sup>, Eric Adechina Alamou<sup>2</sup>

<sup>1</sup>West African Science Service Center on Climate Change and Adapted Land Use, GRP Water Resources, University of Abomey-Calavi, Abomey-Calavi, Benin

<sup>2</sup>Laboratory of Applied Hydrology, National Water Institute, University of Abomey-Calavi, Abomey-Calavi, Benin

## Email address:

biaoeliezer@yahoo.fr (E. I. Biao), ericalamou@yahoo.fr (E. A. Alamou)

\*Corresponding author

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**Abstract:** Nowadays, there is evidence that hydrological processes exhibit long-range dependence (LRD), i.e. power-type decay of autocorrelation also known as the Hurst phenomenon. This means that the stationarity assumption of hydrological time series, which has been widely used in the past, cannot be further advocated. The objective of this paper is to detect the long-range dependence in rainfall in Oueme River basin and to understand how the Hurst coefficient influences the river discharge dynamics. To this end, this paper formulated the Hurst phenomenon that characterized hydrological and other geophysical time series. Then, the fractional generalization of the triple relationship between the fractional Brownian motion, the corresponding stochastic differential equations (SDE) describing the river basin and the deterministic fractional Fokker-Planck equations (FPE) is analysed for the modelling of the river discharge dynamics. This fractional FPE provides an essential tool for the study of the dynamics of the river discharge in Oueme River basin.

**Keywords:** Hurst Coefficient, Fractional Brownian Motion, Stochastic Differential Equations, Fractional Fokker-Planck Equations, Probability Distribution Function

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## 1. Introduction

The last decades have been marked by rapid change of climate on a global scale [1-3]. As a consequence, the uncertainty or unpredictability becomes greater when moving from climatic variables, such as temperature to hydrological variables, such as rainfall and runoff and from the coarse spatial scale of hydrological models [1]. In parallel, the importance of these hydrological variables is greater when dealing with engineering and management issues, such as design and operation of hydro systems. It is therefore important to deal with the uncertainty issue such as uncertainty due to the random character of natural processes governing water supply (rainfall, streamflow, etc.). It should be acknowledged that much of water resources management takes place in an environment in which the basic input information, i.e., rainfall, is not known accurately [4]. In other words, water resources managers and modellers are bound to deal with uncertainty, mostly due to insufficient

data and imperfect knowledge. Good planning and management demand a strong theoretical basis and the proper application of fundamental notions. Traditionally, hydrological statistics, the branch of hydrology that deals with uncertainty, has been based on the implicit assumption of a stable climate. Indeed, pure randomness, where different variables are identically distributed and independent, is sometimes a useful model. However, this disagrees with the fact that climate has changed irregularly on all time scales throughout the history of the Earth, as noted by [1]. The key question that arises is how best to include these hydroclimatic fluctuations in water planning and management.

Nowadays, there is evidence that hydrological processes exhibit long-range dependence (LRD), i.e. power-type decay of autocorrelation also known as the Hurst phenomenon [5]. Indeed, [6] tried to model the river levels over the year as a Brownian motion process. He discovered to his surprise that the river level is not totally random. Instead the process

increments have some vivid correlation, which indicates that the natural phenomena of river level fluctuation follows a biased random walk or Fractional Brownian motion (FBM) path more than that of a regular Brownian motion.

Several authors used the rescaled range analysis (R/S), the modified rescaled range (R/S), the aggregated variance method and the aggregated standard deviation (ASD) method for the hydrological studies [7-9]. They concluded that several other natural processes including lake levels, rainfall, temperature, sunspot counts, and tree rings, etc exhibit long memory. In statistical terms, the presence in a time series of long-term fluctuations implies dramatically increased uncertainty, especially on long time scales, in comparison to classical statistics. This is easy to understand as the observed record could be a small portion of a longer cycle whose characteristics might be difficult to infer on the basis of the available observations. In this respect, in processes characterized by long-range dependence, the results of the statistical analysis may be difficult to decipher. As a consequence, the application of statistical tools to climatic time series should be carefully revisited to locate points that may produce misleading or incorrect results.

The objective of this paper is to detect the long-range dependence in rainfall in Oueme River basin and to

understand how the Hurst coefficient influences the river discharge dynamics. To this end, the Hurst phenomenon, which characterized hydrological and other geophysical time series, is formulated. Moreover, the FBM, which is the most classical process commonly used for a system with long-range dependence, is considered as an approximation of rainfall fluctuation in Oueme River basin. The FBM is useful in modelling anomalous diffusive phenomena and is also a universal model that under some restrictions exhibits super or sub diffusive behavior. Today, the relationship between the stochastic differential equation (SDE) driven by Brownian motion and their associated Fokker-Planck equation (FPE) is well understood [10-11]. The present paper addresses the fractional generalization of this triple relationship between the driving process (i.e., the FBM), the corresponding SDE describing the river basin and the deterministic fractional FPE. Fractional FPE have been used to model the dynamics of complex processes in many fields, including physics, hydrology. Complexity includes phenomena such as weak or strong correlations, different sub or super-diffusive modes, memory and jump effect. Therefore, fractional FPE can be an essential tool for the study of the dynamics of river discharge.

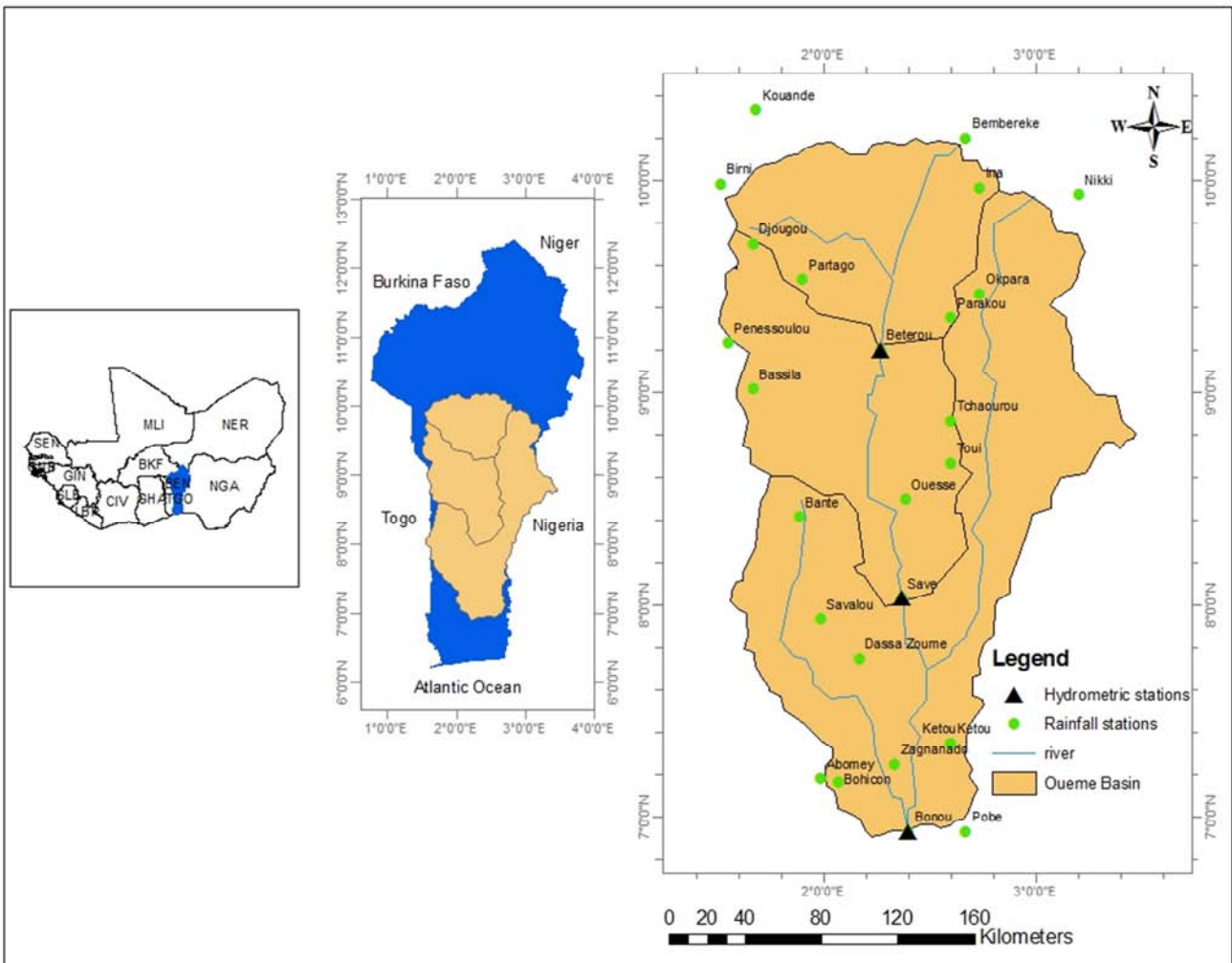


Figure 1. Study area and the hydrometeorological stations used.

## 2. Study Location and Location

The Ouémé catchment at the Bonou outlet extends to 49, 256 km<sup>2</sup> of surface area and makes up of roughly 43% of the Benin country [12] between 6.8 and 10.2 °N of latitude. About 89% of the catchment is located in Benin, about 10% in Nigeria and about 1% in Togo (Figure 1). On a global view, Benin extends from the Niger River to the Atlantic Ocean, with a relatively flat terrain. Benin lies entirely in the tropical Sub-Saharan region with a wet and dry climate. A semi-arid environment is met northwards, made up of savannahs and small mountains (about 600 m), while the south of the country consists of a low coastal plain with marshlands, lakes and lagoons. Meteorological data (daily rainfall data and daily potential evapotranspiration, calculated by the Penman formula) and daily discharge data were provided respectively by the Benin Meteorological Department, ASCENA (Agency for Air Navigation Safety in Africa and Madagascar) and the National Directorate of Water (DG-Eau). A Kruskal-Wallis homogeneity test was conducted on every station of the study area. This test ensures that spatial homogeneity of the rainfall can be assumed. The period 1961 – 2010 was chosen as the study period and spatialized regional daily mean rainfall obtained by kriging method by [11] was used in this paper.

## 3. Model of Long-Range Dependence in Rainfall: Fractional Brownian Motion

In probability theory, fractional Brownian motion (FBM) is a generalization of Brownian motion. Unlike classical Brownian motion, the increments of FBM need not be independent. The FBM was developed specifically to account for the Hurst phenomenon. The connection with Hurst’s law is the parameter H in FBM. FBM is defined as:

$$B^H(t) - B^H(0) = \int_0^t (t-s)^{H-1/2} dB(s), \quad (1)$$

where  $t > 0$  and  $0 < H < 1$ .

The value assigned to H, the Hurst coefficient, determines the range of behavior of FBM. In fact, the long memory effect in a time series is quantified using the Hurst coefficient. The Hurst coefficient is referred to as an index of long-range dependence. Long memory property denotes that a time series has a slowly declining correlogram. The auto covariance function of the increments,  $C(\tau)$ , is defined by:

$$C(\tau) = \langle n(t, h)n(t + \tau, h) \rangle, \quad (2)$$

where  $n(t, h)$  is the increments of FBM represent by  $B^H(t)$ ,

$$n(t, h) = B^H(t+h) - B^H(t). \quad (3)$$

As shown in Appendix A, equation (2) may be expressed in terms of the mean squared increments of  $B^H(t)$ , which for  $\tau \gg h$  ultimately leads to

$$C(\tau) = \sigma^2 h^2 [H(2H-1)] |\tau|^{2H-2}. \quad (4)$$

One can see why the limits of 0 and 1 are imposed on H. Equation (4) defines the classes of correlation:

$H = 1/2 \Rightarrow C(\tau)$  is zero; this indicates a Brownian motion process;

$0 < H < 1/2 \Rightarrow C(\tau)$  is negative. This indicates an anti-persistent process, which means an up value is more likely followed by a down value, and vice versa;

$1/2 < H < 1 \Rightarrow C(\tau)$  is positive. This indicates a persistent process, which means the direction of the next value is more likely the same as current value.

The latter case,  $1/2 < H < 1$ , has been of main interest in surface hydrology because the positive correlation feature can be used to model processes that tend to cluster first on one side of the mean and then the other. These were the types of processes discovered and studied by [6, 13].

High values of H, particularly those approaching 1, indicate enhanced change at large scales or strong clustering of similar values. In others words, in a stochastic framework and in stationary terms, change can be characterized by the Hurst coefficient [14].

## 4. Detecting Long-Range Dependence in Rainfall Dynamics

We analyse the rainfall in the Oueme River basin to check the existence of a long-range dependence in the time series using Hurst coefficient. In the present paper, The Hurst coefficient is calculated by using the rescaled range analysis (R/S analysis) method, the rescaled range (R/S) method modified by [8], the aggregated variance method [9] and the aggregated standard deviation (ASD) method.

### 4.1. The R/S Method

The R/S analysis is the range of partial sums of deviations of a time series from its mean, rescaled by its standard deviation. To study the long-range dependence in the rainfall time series, the following algorithm is used:

A time series  $(X_k)_{k \in 1, N}$  is divided into d sub-series of length m. For each sub-series

$n = 1, \dots, d$ ;

Find the mean,  $E_n$  and the standard deviation,  $S_n$ ;

Normalize the data  $(X_{in})$  by subtracting the sub-series mean:  $Z_{in} = X_{in} - E_n$ ,

$i = 1, \dots, m$ ;

Create a cumulative time series:

$$Y_{in} = \sum_{j=1}^i Z_{jn}, \quad i = 1, \dots, m; \quad (5)$$

$$\text{Find the range } R_n = \max_{j=1,m} Y_{jn} - \min_{j=1,m} Y_{jn}; \quad (6)$$

Rescale the range  $R_n / S_n$ ;

Calculate the mean value of the rescaled range for all sub-series of length  $m$ :

$$(R/S)_m = \frac{1}{d} \sum_{n=1}^d R_n / S_n \quad (7)$$

Hurst found that  $(R/S)$  scales by power – law as  $n$  increases, which indicates

$$(R/S)_n = c.n^H. \quad (8)$$

In practice, in classical  $R/S$  analysis, the Hurst coefficient  $H$  can be estimated as the slope of  $\log / \log$  plot of  $(R/S)_n$  versus  $n$ .

$$\log\left(\frac{R}{S}\right)_n = H \log n + c_1. \quad (9)$$

Various papers, [15-18] emphasized the superiority of the  $R/S$  analysis compared to more traditional methods of detection of long – term persistence such as studying autocorrelation, reports variances and spectral analysis. [19] showed that the  $R/S$  analysis can detect the presence of long – term persistence even in a highly non-Gaussian time series.

#### 4.2. Modified R/S Method

[8] proposed another statistic, called "modified  $R/S$  statistic." Its limit distribution is invariant to different forms of short memory processes. This method allows testing the null hypothesis of no long term persistent (LTP) against the alternative of STP. The modified  $R/S$  statistic has the following form:

$$\tilde{Q}_q(n) = \frac{1}{\sqrt{n}} \frac{R(n)}{S_q(n)} \quad (10)$$

with

$$S_q(n) = \left\{ S_n^2 + \frac{2}{n} \sum_{j=1}^q w_j(q) \left[ \sum_{i=j+1}^n (X_i - \bar{X}_n)(X_{i-j} - \bar{X}_n) \right] \right\}^{1/2} \quad (11)$$

$S_n^2$  and  $\bar{X}_n$  are respectively the empirical variance and mean  $w_j(q) = 1 - \frac{j}{q+1}$

( $q=1, 2, \dots, q$ ) are the weights proposed by [20]. In practice, the selection of the integer  $q$  is a real problem. [21-22] have shown by Monte Carlo studies that when  $q$  is relatively large compared to the sample size, the estimator is biased and therefore  $q$  must be chosen as a small integer,

while other studies have shown by Monte Carlo that  $q = 1$  is an acceptable choice. Then, we choose  $q = 1$ . Contrary to the classic  $R/S$  statistic, the limit distribution of the modified  $R/S$  is known and the statistic  $V$  defined by:

$$V = \frac{\tilde{Q}_T}{\sqrt{T}}, \quad (12)$$

converges to the extent of a Brownian bridge on the unit interval.

#### 4.3. Aggregated Variance Method

The method of aggregated variance is based on the aggregation of the time series into several blocks  $Y_k^{(m)}$  of size  $m$ :

$$Y_k^{(m)} = \frac{1}{m} \sum_{t=1+m(k-1)}^{km} Y(t) \quad (13)$$

$k = 1, 2, \dots, T/m$  is the block sequence number.  $T$  is the number of observations.

Different values are selected for the parameter  $m$ ,  $\{m_i, i \geq 1\}$  such as  $\frac{m_{i+1}}{m_i} = C$

where  $C$  is a constant that depends only on the lengths of the time series and the desired number of points. Then, one calculates the variance  $V(Y^{(m)})$  of  $Y^{(m)}$ :

$$V(Y^{(m)}) = \frac{1}{T/m} \sum_{k=1}^{T/m} (Y_k^{(m)})^2 - \left[ \frac{1}{T/m} \sum_{k=1}^{T/m} (Y_k^{(m)}) \right]^2 \quad (14)$$

The procedure is repeated for successive values of  $m$  and one has  $V(Y^{(m)}) \approx C.m^{2H-2}$  where  $C$  is a constant.

A regression of  $\log V(Y^{(m)})$  on  $\log(m)$  is a straight line with slope  $2H-2$ , which provides an estimator of  $H$ .

#### 4.4. Aggregated Standard Deviation

To apply the aggregated standard deviation (ASD) method, we need to assess the standard deviation at several time scales.

Let  $X_i$  be a stationary process on discrete time  $i$  (referring to days here) with standard deviation  $\sigma$  and let

$$X_i^{(k)} = (X_i + \dots + X_{i-k+1}) / k \quad (15)$$

be the aggregated process at time scale  $k$ , with standard deviation  $\sigma^{(k)}$ . The long term persistent is expressed by elementary scaling process:

$$\sigma^{(k)} = \frac{\sigma}{k^{1-H}} \quad (16)$$

Equation (16) corresponds to a stochastic process in discrete time and termed Hurst-Kolmogorov process (HK). Its continuous time form is [14]

$$\sigma^{(k)} = \sigma^{(a)} \left( \frac{a}{k} \right)^{1-H} \quad (17)$$

where  $a$  is any time scale,  $\sigma(a)$  is the standard deviation at scale  $a$ , and both  $a$  and  $k$  have units of time. To determine  $H$ , we use the algorithm constructed by [1].

Table 1 presents the results of the estimation of  $H$  using the four methods described above.

**Table 1.** Hurst coefficient for rainfall time series in Oueme at Bétéro, Save and Bonou sub-catchments.

Sub-catchments	R/S	Modified R/S Variance	Aggregated method	ASD
Bétéro	$H = 0.69$	$H = 0.67$	$H = 0.66$	$H = 0.64$
Save	$H = 0.68$	$H = 0.67$	$H = 0.64$	$H = 0.65$
Bonou	$H = 0.70$	$H = 0.68$	$H = 0.64$	$H = 0.66$

The obtained Hurst coefficient values for rainfall shows that all three series, for the investigated methods, exhibit persistent behaviour. These values of  $H$  also suggest the existence of long-range memory in the rainfall dynamics. The fact that  $1/2 < H < 1$ , means that the rainfall process has positively correlated increments. These findings are in accordance with the types of processes discovered and studied by [6, 13]. Furthermore, these results are consistent with the works of [23] which used a simple scaling stochastic (SSS) process to show the evidence of the hydroclimatic fluctuations on multiple time scales, a behaviour that is none other than the Hurst phenomenon. In fact, the Hurst phenomenon is a manifestation of irregular climate fluctuations on several scales. Therefore, identifying the Hurst phenomenon on a specific hydroclimatic time series provides some indications of climate fluctuations

To consolidate the results obtained above, let us plot the covariance of the increments,  $C(\tau)$ , given by equation 4. To this end, letting  $h = \sigma = 1$  and using the estimated Hurst coefficient for Bétéro, Save and Bonou yields

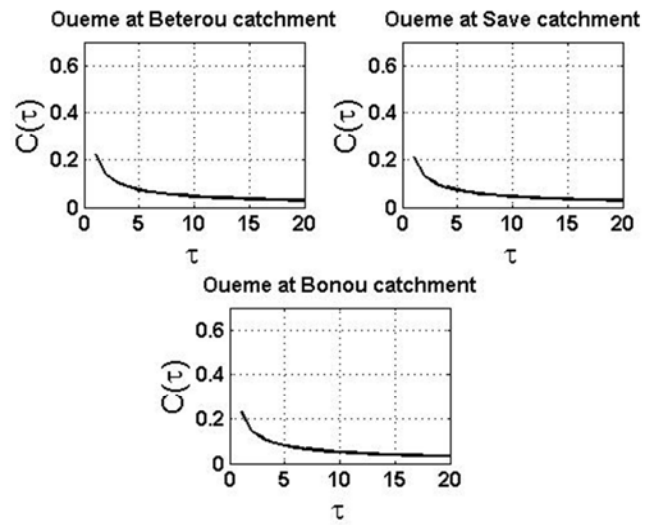
$$C(\tau) = \frac{0.22}{|\tau|^{0.68}} \text{ in Bétéro,} \quad (18)$$

$$C(\tau) = \frac{0.21}{|\tau|^{0.67}} \text{ in Save,} \quad (19)$$

$$C(\tau) = \frac{0.23}{|\tau|^{0.66}} \text{ in Bonou,} \quad (20)$$

a positive correlation that fall off relatively slowly as  $\tau$  gets large, as illustrated in Figure 2. Such long-range correlations in the rainfall and hydrologic records have been difficult to explain but appear real [24]. In such a case, Fractional Brownian motion is a good choice for the modelling the random component of rainfall in Oueme River basin. Figure 2 shows the plot of dimensionless covariance of increments as a function of  $\tau$  in Oueme at Bétéro, Save and Bonou sub-

catchments.



**Figure 2.** Plot of dimensionless covariance of increments as a function of  $\tau$  in Oueme at Bétéro, Save and Bonou sub-catchments. Notice the long tail of the plot that is essential to preserving the Hurst effect.

## 5. Modelling River Discharge Dynamics

### 5.1. Fractional Generalization of the Triple Relationship Between the FBM, the Corresponding SDE and the Associate FPE

The time-varying behaviour of hydrological systems can be described by ordinary, deterministic differential equation (21)

$$\frac{dQ}{dt} = f(Q, t) \quad (21)$$

where  $Q$  is the discharge at the outlet of the river basin and  $f(Q, t)$  is a function completely describing the dynamics of the system. However, this model may not be perfect because there are uncertainties in the system. In such cases, a stochastic extension of the model is preferred, such that the hydrological system behaviour is described in terms of probabilities. When the model is extended by taking into account the uncertainties, equation (21) changes to

$$\frac{dQ}{dt} = f(Q, t) + g(Q, t)\varepsilon(t), \quad (22)$$

where  $\varepsilon(t)$  is the noise or fluctuation, i.e., the uncertainties related to the random component of rainfall and  $g(Q, t)$  is a function specifying the amount of noise. Here, the function  $f(Q, t)$  is derived from the Hydrological Model based on the Least Action Principle (HyMoLAP). The HyMoLAP uses the principle of minimum energy expenditure and has two physical parameters  $\nu$  and  $\lambda$ . The parameter  $\nu$  describes the non-linearity of the transformation of rainfall into runoff, while the parameter  $\lambda$  describes properties related to the geomorphology and pedology of the catchment. This model

has been used in a number of studies in the past. Inputs of the model include area catchment rainfall and potential evapotranspiration, while the output is the estimated river discharge. The HyMoLAP has been discussed extensively in many previous papers [11, 25-28]. Details of the model can be found therein.

The driving process of a stochastic differential equation plays a key role in the dynamics and future evolution of the solution to that SDE. Thus, in this paper, the fluctuation  $\varepsilon(t)$  is modelled by FBM. Thus, the SDE describing the river basin is given by

$$dQ(t) = f(Q(t), t)dt + g(Q(t), t)dB^H(t), \quad (23)$$

where  $f(Q, t) = -\frac{\nu}{\lambda}Q^{2\nu-1} + \psi(q, t)$ ;  $\psi$  describes the model input ( $q$  is the effective rainfall);  $g(Q(t), t) = \sigma^2$  is the standard deviation of  $f(Q(t), t)$ .

The FBM-driven FPE with respect to equation (23) is given by equation (24). Its derivation is given in Appendix B.

$$\begin{aligned} \frac{\partial P(Q, t)}{\partial t} + \frac{\partial}{\partial Q} [f(Q(t), t)P(Q, t)] - \\ \frac{\partial^2}{\partial Q^2} \left[ g(Q(t), t) \int_0^t g(Q(s), s) \varphi(s, t) ds P(Q, t) \right] = 0 \end{aligned} \quad (24)$$

where the function  $\varphi(s, t)$  is defined by

$$\varphi(s, t) = H(2H-1) |s-t|^{2H-2}. \quad (25)$$

[27] showed the effect of different specific noises on the dynamics of river discharge by comparing their associated FPE. They found that each specific type of noises modifies the form of FPE. The result given by equation (24) confirms these findings since the derived FBM – FPE is not similar to the ordinary FPE driving by Brownian motion. As a consequence, the dynamics of the river discharge for FBM could not be the same as the one derived from the ordinary FPE.

## 5.2. Derivation of the Time-Dependent Probability Distributions Function of the River Discharge

Now we focus on the evolution of the probability distribution function of system (23). For simplicity, one can set  $\nu = 1$ ,  $\theta = \nu / \lambda$ . According to the FBM-driven FPE (24), one has

$$\begin{aligned} \frac{\partial P(Q, t)}{\partial t} + \frac{\partial}{\partial Q} [(-\theta Q(t) + \psi(q, t)P(Q, t))] - \\ \frac{\partial^2}{\partial Q^2} [\sigma^2 H t^{2H-1} P(Q, t)] = 0 \end{aligned} \quad (26)$$

which leads to

$$\begin{aligned} \frac{\partial P(Q, t)}{\partial t} - \theta \frac{\partial}{\partial Q} (Q(t)P(Q, t)) + \\ \frac{\partial}{\partial Q} (\psi(q, t)P(Q, t)) - H\sigma^2 t^{2H-1} \frac{\partial^2 P(Q, t)}{\partial Q^2} \end{aligned} \quad (27)$$

with initial condition  $P(Q, 0) = \delta(Q - Q_0)$ . Denote the Fourier transform with respect to  $Q$ , for each fixed  $t$ , of  $P(Q, t)$  by

$$\hat{P}(\xi, t) = F\{P(Q, t)\} = \int_{-\infty}^{\infty} h(\omega, t) e^{-i\xi\omega} d\omega. \quad (28)$$

So applying the Fourier transform to both sides of equation (27) gives

$$\frac{\partial \hat{P}(\xi, t)}{\partial t} + (\theta\xi + \psi) \hat{P}(\xi, t) = -H\sigma^2 t^{2H-1} \xi^2 \hat{P}(\xi, t) \quad (29)$$

By the method of characteristics, we suppose that

$$\begin{aligned} \frac{d\hat{P}(\xi, t)}{dt} = \frac{\partial \hat{P}(\xi, t)}{\partial t} + \frac{\partial \hat{P}(\xi, t)}{\partial \xi} \frac{d\xi}{dt} = \\ \frac{\partial \hat{P}(\xi, t)}{\partial t} + (\theta\xi + \psi) \frac{\partial \hat{P}(\xi, t)}{\partial \xi} \end{aligned} \quad (30)$$

which implies that  $\frac{d\xi}{dt} = \theta\xi + \psi$ , and its solution is expressed by  $\xi = \xi_0 e^{\theta t}$ , with  $\xi_0 = \int \psi e^{-\theta t}$ . It means that the left hand side of equation (29) is the total time derivative along each of the curve  $\xi(t, t_0)$ . On this curve, we then get

$$\begin{aligned} \frac{d\hat{P}(\xi, t)}{dt} = -H\sigma^2 t^{2H-1} \xi^2 \hat{P}(\xi, t) = \\ -H\sigma^2 \xi_0^2 t^{2H-1} e^{2\theta t} \hat{P}(\xi, t) \end{aligned} \quad (31)$$

Since  $\hat{P}(\xi, 0) = F\{\delta(Q - Q_0)\} = e^{-\xi Q_0}$ , we further obtain

$$\begin{aligned} \hat{P}(\xi, t) = \hat{P}(\xi_0, 0) \exp \left( \int_0^t -H\sigma^2 \xi_0^2 s^{2H-1} e^{2\theta s} ds \right) \\ = \exp \left[ -i\xi(Q_0 e^{-\theta t}) - \frac{\xi^2}{2} \left( 2H\sigma^2 e^{-2\theta t} \int_0^t s^{2H-1} e^{2\theta s} ds \right) \right]. \end{aligned} \quad (32)$$

Let  $\mu(t) = Q_0 e^{-\theta t}$  and  $\gamma^2(t) = 2H\sigma^2 e^{-2\theta t} \int_0^t s^{2H-1} e^{2\theta s} ds$ , and it follows that

$$P(Q, t) = \frac{1}{\sqrt{2\pi\gamma^2(t)}} \exp \left( -\frac{(Q - \mu(t))^2}{2\gamma^2(t)} \right). \quad (33)$$

This means that the derived time-dependent probability distribution of the river discharge  $Q$  is a Gaussian process



with the mean  $\mu(t)$  and a standard deviation  $\gamma(t)$ . In most studies using HyMoLAP, the parameter  $\lambda$  is high. Thus,  $\theta = 1/\lambda$  tends to 0. In such a case,  $\mu(t) = Q_0$ ,  $\gamma^2(t) = \sigma^2 t^{2H}$  and the time-dependent probability distribution (33) gives

$$P(Q, t) = \frac{1}{\sqrt{2\pi\sigma^2 t^{2H}}} \exp\left(-\frac{(Q-Q_0)^2}{2\sigma^2 t^{2H}}\right). \quad (34)$$

By letting  $u = \frac{Q-Q_0}{\sigma}$ , one derive

$$P(u, t) = \frac{1}{\sqrt{2\pi\sigma^2 t^{2H}}} \exp\left(-\frac{u^2}{2t^{2H}}\right). \quad (35)$$

We carry out the numerical simulations for evolution of the probability distribution function. Figures 3, 4 and 5 show the time-dependent probability distribution  $P(u, t)$  of the standardized daily discharge  $u$  over the investigated sub-catchments.

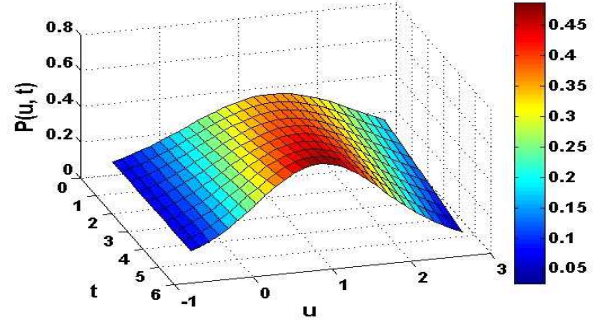
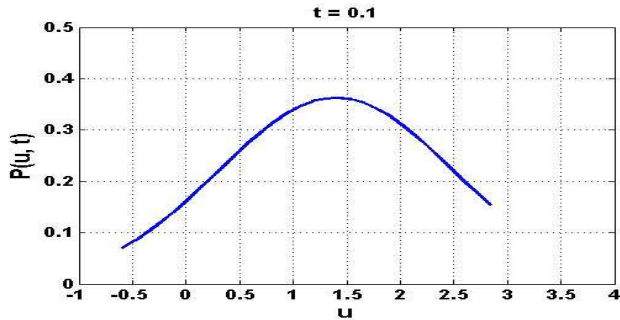


Figure 3. Time-dependent probability distribution  $P(u, t)$  of the standardized daily discharge  $u$  for the Bétérou sub-catchment for the period 1961–2010. The colours toward the blue end of the colour map indicate low probability and the colours at the red end of the colour map indicate high probability.

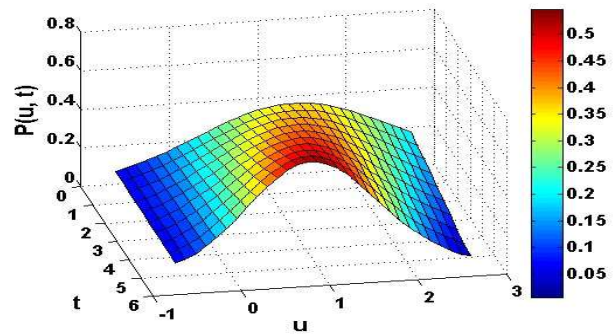
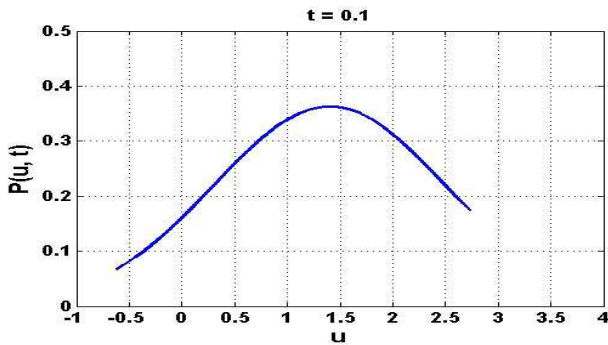


Figure 4. Time-dependent probability distribution  $P(u, t)$  of the standardized daily discharge  $u$  for the Save sub-catchment for the period 1961–2010.

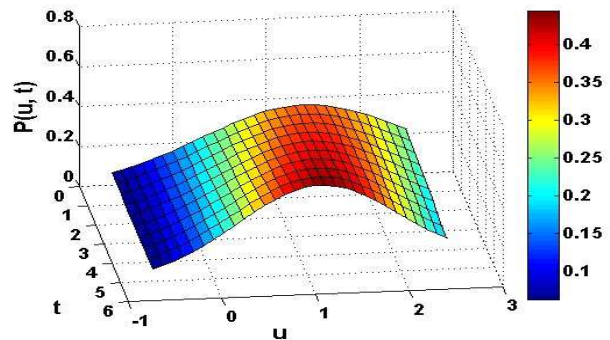
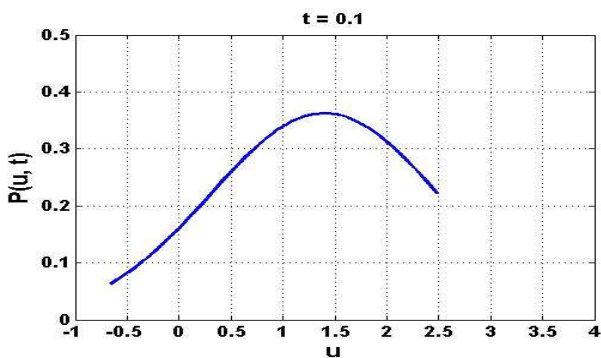


Figure 5. Time-dependent probability distribution  $P(u, t)$  of the standardized daily discharge  $u$  for the Bonou sub-catchment for the period 1961–2010.

It can also be seen from these figures that the probability distribution function of the river discharge has decreasing tail area with time. This illustrates the significant and distinguishing influence of Hurst parameter  $H$  on the dynamics of the river discharge when time  $t$  evolves in comparison to the works of [11] which are based on the

assumption that statistical samples consist of independent, identically, distributed variables (e.g. Gaussian white noise). The probability distribution function of the solution  $Q_t$  carries significant dynamical information. The analyses in this study act as a warning that the classical hydrological statistics describes only a portion of the natural uncertainty of

hydroclimatic processes, because it is based on the implicit assumption of a stable climate. In addition, its use may characterize a regular behaviour of hydroclimatic processes as an unusual phenomenon.

## 6. Conclusion

The main contribution of this paper is to detect the long-range dependence in rainfall in Oueme River basin and to understand how the Hurst coefficient influences the river discharge dynamics. The achievement of this analysis stemmed first from the estimation of Hurst coefficient by using different methods in order to detect long-range dependence in rainfall dynamics, and second from the fractional generalization of this triple relationship between the driving process (i.e., the FBM), the corresponding SDE describing the river basin and the deterministic fractional order FPE for the modelling of the river discharge dynamics. The results of the estimation of the Hurst coefficient, in Oueme River basin, showed the existence of long-range memory in the rainfall dynamics. The fractional FPE provides an essential tool for the study of the dynamics of various complex processes arising in anomalous diffusion in hydrology.

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## Appendix A: Auto Covariance Function of the Increments, $C(\tau)$

$$C(\tau) = \langle n(t, h)n(t + \tau, h) \rangle \quad (A1)$$

$$\begin{aligned} &= [B^H(t+h) - B^H(t)][B^H(t+h+\tau) - B^H(t+\tau)] \\ &= \frac{1}{2} \langle [B^H(t+h) - B^H(t+\tau)]^2 + [B^H(t) - B^H(t+h+\tau)]^2 \\ &\quad - [B^H(t+h) - B^H(t+h+\tau)]^2 - [B^H(t) - B^H(t+\tau)]^2 \rangle \end{aligned}$$

The self-affinity of FBM allows (A1) to be written as

$$C(\tau) = \tau |^{2H} \frac{\sigma^2}{2} \left[ \left| 1 - \frac{h}{\tau} \right|^{2H} + \left| 1 + \frac{h}{\tau} \right|^{2H} - 2 \right]. \quad (A2)$$

For  $h/\tau \leq 1$  one may use a Taylor series to write

$$\left( 1 \pm \frac{h}{\tau} \right)^{2H} \approx 1 \pm \frac{2Hh}{\tau} + \frac{H(2H-1)h^2}{\tau^2} \quad (A3)$$

Combining (A2) and (A3) yields equation (4), that is

$$C(\tau) = \sigma^2 h^2 [H(2H-1)] |\tau|^{2H-2}.$$

## Appendix B: Derivation of the FBM-driven Fokker-Planck Equation

Let consider the SDE describing the river basin

$$dQ(t) = f(Q(t), t)dt + g(Q(t), t)dB^H(t) \quad (B1)$$

Then, applying the fractional Ito formula [29] to the process  $h(Q)$  gives

$$\begin{aligned} dh(Q) = & \left( \frac{dh(Q)}{dQ} f(Q(t), t) + \right. \\ & \left. \frac{d^2 h(Q)}{dQ^2} g(Q(t), t) \int_0^t g(Q(s), s) \varphi(s, t) ds \right) dt \\ & + \frac{dh(Q)}{dQ} g(Q(t), t) dB^H(t) \end{aligned} \quad (B2)$$

where the function  $\varphi(s, t)$  is defined by

$$\varphi(s, t) = H(2H-1) |s-t|^{2H-2} \quad (B3)$$

Taking the expectation of equation (B2), one has

$$\begin{aligned} E[dh(Q)] = & E \left[ \frac{dh(Q)}{dQ} f(Q(t), t) + \right. \\ & \left. \frac{d^2 h(Q)}{dQ^2} g(Q(t), t) \int_0^t g(Q(s), s) \varphi(s, t) ds \right] dt \\ & + E \left[ \frac{dh(Q)}{dQ} g(Q(t), t) dB^H(t) \right] \end{aligned} \quad (B4)$$

Therefore, we get

$$E \left[ \frac{dh(Q)}{dt} \right] = E \left[ \frac{dh(Q)}{dQ} f(Q(t), t) + \frac{d^2 h(Q)}{dQ^2} g(Q(t), t) \int_0^t g(Q(s), s) \varphi(s, t) ds \right] \quad (B5)$$

Denote by  $P(Q, t)$  the time-dependent probability distribution of the river discharge, we have

$$E[h(Q)] = \int_{-\infty}^{\infty} h(Q) P(Q, t) dQ, \quad (B6)$$

which implies that  $E \left[ \frac{dh(Q)}{dt} \right] = \int_{-\infty}^{\infty} h(Q) \frac{\partial P(Q, t)}{\partial t} dQ$ . Thus, we further get



$$\int_{-\infty}^{\infty} h(Q) \frac{\partial P(Q,t)}{\partial t} dQ = \int_{-\infty}^{\infty} \left( \frac{dh(Q)}{dQ} f(Q(t),t) + \frac{d^2 h(Q)}{dQ^2} g(Q(t),t) \int_0^t g(Q(s),s) \phi(s,t) ds \right) P(Q,t) dQ. \quad (B7)$$

After the integration by parts we find that

$$\int_{-\infty}^{\infty} \frac{dh(Q)}{dQ} f(Q(t),t) P(Q,t) dQ = \int_{-\infty}^{\infty} h(Q) \frac{\partial (f(Q(t),t) P(Q,t))}{\partial Q} dQ \quad (B8)$$

since  $P(\pm\infty, t) = 0$ . In a similar way, we further have

$$\int_{-\infty}^{\infty} \frac{d^2 h(Q)}{dQ^2} g(Q(t),t) \int_0^t g(Q(s),s) \phi(s,t) ds P(Q,t) dQ = \int_{-\infty}^{\infty} h(Q) \frac{\partial^2 \left( g(Q(t),t) \int_0^t g(Q(s),s) \phi(s,t) ds P(Q,t) \right)}{\partial Q^2} dQ. \quad (B9)$$

From equation (B7), (B8) and (B9), we obtain

$$\int_{-\infty}^{\infty} h(Q) \left( \frac{\partial P(Q,t)}{\partial t} + \frac{\partial (f(Q(t),t) P(Q,t))}{\partial Q} - \frac{\partial^2 \left( g(Q(t),t) \int_0^t g(Q(s),s) \phi(s,t) ds P(Q,t) \right)}{\partial Q^2} \right) dQ = 0, \quad (B10)$$

which leads to equation (24), that is

$$\frac{\partial P(Q,t)}{\partial t} + \frac{\partial}{\partial Q} [f(Q(t),t) P(Q,t)] - \frac{\partial^2}{\partial Q^2} \left[ g(Q(t),t) \int_0^t g(Q(s),s) \phi(s,t) ds P(Q,t) \right] = 0$$

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